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The tangent functor category revisited

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Let $\mathscr{M}$ be the category of C^∞-manifolds, and let $T: \mathscr{M} \to \mathscr{M}$ be the tangent functor assigning to each manifold its tangent bundle. The tangent functor category is the category whose objects are the iterated tangent functors $T^n: \mathscr{M} \to \mathscr{M}$, $n \geq 0$, and whose morphisms are the natural transformations. This tangent functor category was studied in Świerczkowski (1974). One of his results was that the tangent functor category is exactly the same as the one described by having the functors $T^n: \mathscr{M} \to \text{Sets}$ assigning to a manifold the *set* of points of its n^{th} iterated tangent bundle, and natural transformations as morphisms. In other words,

(1) if $\lambda: T^n \dot{\longrightarrow} T^m$ is a natural transformation whose components λ_M are a priori given as arbitrary functions $\lambda_M: T^n(M) \to T^m(M)$, then it follows that all λ_M are smooth maps.

We will denote this category of functors $T^n: \mathscr{M} \to \text{Sets}$ by $\mathscr{T}$. The main result of Świerczkowski was an explicit description of $\operatorname{Hom}_{\mathscr{T}}(T^n, T^m)$:

(2) a morphism $T^n \dot{\longrightarrow} T^r$ corresponds uniquely to an n-tuple of finite families $A^i = \{a^i_H\}_H$ of real numbers, indexed by the nonempty subsets $H \subseteq \{1, \ldots, r\}$, which satisfy the following condition for every $i = 1, \ldots, n$: For every nonempty $H \subset \{1, \ldots, r\}$, the sum

$$\sum \{a^i_{H_1} \cdot a^i_{H_2} | H \text{ is the disjoint union of } H_1 \text{ and } H_2,\ H_1 \neq \emptyset \neq H_2\} \text{ is zero.}$$

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From this 'classification theorem' a number of corollaries were derived: (3) the 'coproduct theorem' telling us that T^n is the coproduct of n copies of T in the tangent functor category, (4) the fact that there is only one monad on the category $\mathscr{M}$ of manifolds having T as underlying functor, and (5) the observation that the group $\mathrm{Aut}_{\mathscr{T}}(T^n, T^n)$ of isomorphisms from T^n to T^n is a Lie-group.

Now if there were a manifold D such that $T^n: \mathscr{M} \to \mathrm{Sets}$ was of the form $T^n(M) = \mathscr{M}(D^n, M) =$ the set of smooth maps from D^n to M, then by the Yoneda lemma $\mathscr{T}(T^n, T^m) \cong \mathscr{M}(D^m, D^n)$, so the problem of classifying $\mathscr{T}(T^n, T^m)$ would be reduced to the analysis of smooth maps $D^m \to D^n$. (Note that results (1) and (3) would then be immediate.)

Unfortunately, there is no such manifold D in $\mathscr{M}$. In other words, the category $\mathscr{M}$ is too small for the study of the tangent functor category. This lack of elbowroom in $\mathscr{M}$ corresponds precisely to the fact that $\mathscr{M}$ does not have the right pullbacks in general (only transversal ones are good). The point that we want to make in this note is that by adjoining finite inverse limits we can enlarge $\mathscr{M}$ to a bigger category, called $\mathscr{F}$, *without* changing the tangent functor category; i.e. $\mathscr{T}$ can equivalently be described as the category of functors $T^n: \mathscr{F} \to \mathrm{Sets}$ and natural transformations between them. This bigger category $\mathscr{F}$ indeed contains such an object D with the property that

$$T^n = \mathscr{F}(D^n, -): \mathscr{F} \to \mathrm{Sets},$$

and the Yoneda lemma now makes the results (1)–(4) mentioned above completely trivial. The result (5) saying that each group $\mathrm{Aut}_{\mathscr{T}}(T^n, T^n)$ is a Lie-group is proved by modifying the classification theorem (2) above so as to get a matrix representation of $\mathrm{Aut}_{\mathscr{T}}(T^n, T^n)$. This is a purely combinatorial argument which is insensitive to the passage from $\mathscr{M}$ to $\mathscr{F}$. So as far as the proof of (5) is concerned, we have nothing to add to Świerczkowski's argument. However, a more general result can in fact be proved in a very simple way, as will be pointed out in the concluding remarks.

We will now first describe this finite inverse limit completion $\mathscr{F}$ of $\mathscr{M}$ and show that the tangent functor category remains the same; the results (1)–(4) will then follow immediately.

$\mathscr{F}$ is the opposite category of the category $(C^\infty\text{-rings})_{fp}$ of finitely presented C^∞-rings, described in Dubuc (1981), Moerdijk and Reyes (1983), and elsewhere. We repeat its definition here. A *C^∞-ring* is an $\mathbb{R}$-algebra A in which one can interpret all smooth maps $\mathbb{R}^n \to \mathbb{R}^m$ $(n, m \geq 0)$ (not just all polynomials, as in the definition of an $\mathbb{R}$-algebra), i.e. for each smooth $f: \mathbb{R}^n \to \mathbb{R}^m$ there is an ($\mathbb{R}$-algebra) homomorphism $A^n \xrightarrow{A(f)} A^m$, and this assignment $f \longmapsto A(f)$ respects composition and identity maps. A homomorphism $A \xrightarrow{\varphi} B$ of C^∞-rings is a ring-homomorphism which commutes with this assignment, i.e. $\varphi^m \circ A(f) = B(f) \circ \varphi^n$. The free C^∞-ring on n generators is the ring $C^\infty(\mathbb{R}^n)$ of smooth functions $\mathbb{R}^n \to \mathbb{R}$. A C^∞-ring A is finitely presented if it is of the form $A = C^\infty(\mathbb{R}^n)/I$ for a finitely generated ideal I. A homomorphism $C^\infty(\mathbb{R}^n)/I \to$

$\to C^\infty(\mathbb{R}^m)/J$ corresponds to an equivalence class of smooth maps $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}^n$ with the property that $I \subseteq \varphi^*(J) = \{f \mid f \circ \varphi \in J\}$, two such maps φ and φ' being equivalent if for each projection $\pi_k: \mathbb{R}^n \to \mathbb{R}$, $\pi_k \circ \varphi = \pi_k \circ \varphi'$ mod J. If A is a finitely presented C^∞-ring, its 'geometric' dual, which is an object of $\mathscr{F}$, is denoted by $\bar{A}$.

With every manifold M we can associate the C^∞-ring $C^\infty(M)$ of smooth functions $M \to \mathbb{R}$, and for a smooth $f: M \to N$ we have a C^∞-ring homomorphism 'compose with f': $C^\infty(N) \to C^\infty(M)$. This defines a (contravariant) functor from manifolds to C^∞-rings. Now every C^∞-ring of this form $C^\infty(M)$ is finitely presented. Indeed, if $M = U$ is open $\subseteq \mathbb{R}^n$, then $C^\infty(U) = C\,(\mathbb{R}^n \times \mathbb{R})/(y \cdot \chi_U(x) - 1)$, where $\chi_U^{-1}(0) = \mathbb{R}^n \setminus U$. But every manifold M is a retract of an open $U \subseteq \mathbb{R}^n$, by the ε-Neighbourhood Theorem, (Guillemin and Pollack (1974), p. 69–70), and retracts of finitely presented rings are finitely presented. Thus we obtain a full embedding functor

$$\mathscr{M} \hookrightarrow \mathscr{F},$$

and we will just write M for $\overline{C^\infty(M)} \in \mathscr{F}$.

Note that $\mathscr{F}$ has finite inverse limits. For example, products are described by

$$\overline{C^\infty(\mathbb{R}^n)/I} \times \overline{C^\infty(\mathbb{R}^m)/J} = \overline{C^\infty(\mathbb{R}^n \times \mathbb{R}^m)/(I, J)}$$

where (I, J) is the ideal generated by $I \circ \pi_1 \cup J \circ \pi_2$. Also observe that each object $\bar{A}$ of $\mathscr{F}$, $A = C^\infty(\mathbb{R}^n)/(f_1, \ldots, f_p)$, is the equalizer of

$$\overline{C^\infty(\mathbb{R}^n)} \underset{0}{\overset{(f_1, \ldots, f_p)}{\rightrightarrows}} C^\infty(\mathbb{R}^p),$$

so it is clear that $\mathscr{F}$ is obtained from $\mathscr{M}$ by adjoining finite limits.

The specific object D of $\mathscr{F}$ that we want to consider is the dual of the C^∞-ring $C^\infty(\mathbb{R})/(x^2)$. By Hadamard's lemma, it is easily seen that maps $D \to M$ in $\mathscr{F}$ correspond to points on the tangent bundle TM of M. In fact, we can define the *tangent bundle* $T(\bar{A})$ of any object $\bar{A}$ of $\mathscr{F}$, $A = C^\infty(\mathbb{R}^n)/I$, by

$$T(\bar{A}) = \bar{A}^D = C^\infty(\mathbb{R}^n \times \mathbb{R}^n)/\left(I, \left\{ \sum_{i=1}^n y_i \frac{\partial f}{\partial x_i} \,\middle|\, f \in I \right\}\right)$$

(where a point of $\mathbb{R}^n \times \mathbb{R}^n$ is written as (x, y)). The 'function-space' notation $\bar{A}^D$ is justified, since if $\bar{B}$ is another object of $\mathscr{F}$, there is a unique *'exponential'* correspondence between maps $\bar{B} \to \bar{A}^D$ and maps $\bar{B} \times D \to \bar{A}$ in $\mathscr{F}$, again by Hadamard's lemma. (Indeed, if A is as above and $B = C^\infty(\mathbb{R}^m)/J$, maps $\bar{B} \to \bar{A}^D$ are represented by smooth $\varphi(z): \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n$, and maps $\bar{B} \times D \to \bar{A}$ by smooth $\psi(z, x): \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$, and the correspondence is given by $\psi(z, x) = \pi_1\varphi(z) + x \cdot \pi_2\varphi(z)$, $\varphi(z) = \langle \psi(z, 0), x \cdot (\partial\psi/\partial x)(z, 0) \rangle$, modulo the ideals involved.)

A *point* of an object $\bar{A}$ is a map $1 \to \bar{A}$ in $\mathscr{F}$, where 1 is the onepoint manifold $(= C^\infty(\mathbb{R}^0))$, and we denote the set of points of $\bar{A}$ by $pt(\bar{A})$. (So if M is a manifold, $pt(M)$ is the underlying set of M.) Thus the functor

$$pt \circ T: \mathscr{F} \to \text{Sets}$$

assigning to each C^∞-ring $\bar{A}$ the set of points of $T(\bar{A})$ extends the tangent functor $T: \mathcal{M} \to \text{Sets}$. But $pt \circ T: \mathcal{F} \to \text{Sets}$ is the functor

$$\operatorname{Hom}_{\mathcal{F}}(D, -): \mathcal{F} \to \text{Sets}$$

by exponential correspondence. Similarly, $T^n: \mathcal{M} \to \text{Sets}$ is extended by $\operatorname{Hom}_{\mathcal{F}}(D^n, -): \mathcal{F} \to \text{Sets}$. Now if λ is a natural transformation $pt \circ T^n \overset{\cdot}{\longrightarrow} pt \circ T^m$, where $pt \circ T^n, pt \circ T^m: \mathcal{M} \to \text{Sets}$, it is not hard to see that λ has a unique extension to a natural transformation of the extensions $pt \circ T^n$ $pt \circ T^m: \mathcal{F} \to \text{Sets}$, essentially since every object of $\mathcal{F}$ is an equalizer of manifolds, as remarked above. But by the Yoneda lemma, a natural transformation $\lambda: \operatorname{Hom}_{\mathcal{F}}(D^n, -) \to \operatorname{Hom}_{\mathcal{F}}(D^m, -)$ corresponds uniquely to a map $D^m \to D^n$ in $\mathcal{F}$. Thus we have proved:

THEOREM. *The tangent functor category may equivalently be described as the dual of the full subcategory of $\mathcal{F}$ whose objects are the 'infinitesimal manifolds' D^m: the object T^n of the former corresponds to the object D^n of the latter, and natural transformations $T^n \overset{\cdot}{\longrightarrow} T^m$ correspond to $\mathcal{F}$-maps $D^m \to D^n$.*

Now the results (1)-(4) above follow easily. For (1) we note that if a natural transformation $\lambda: T^n \overset{\cdot}{\longrightarrow} T^m$ comes from a map $\varphi: D^m \to D^n$ in $\mathcal{F}$, its components $\lambda_M: T^n(M) \to T^m(M)$, i.e. $M^{D^n} \to M^{D^m}$ come from 'composition with φ' by the exponential correspondence, i.e. are maps in $\mathcal{F}$, hence smooth. The coproduct theorem (3) is trivial, now. The classification theorem (2) is an immediate application of Hadamard's lemma: a map $D^m \to D^n$ is an n-tuple of maps $\varphi: D^m \to D$, i.e.

$$\varphi: \overline{C^\infty(\mathbb{R}^m)/(x_1^2, \ldots, x_m^2)} \to \overline{C^\infty(\mathbb{R})/(y^2)}$$

which are represented by smooth $\varphi: \mathbb{R}^m \to \mathbb{R}$. Now apply Hadamard and write

$$\varphi(x_1, \ldots, x_m) = \varphi(0, \ldots, 0) + x_1 \frac{\partial \varphi}{\partial x_1}(x_1, \ldots, x_m) + x_1^2 \psi(x_1, \ldots, x_m),$$

so φ is equivalent to $\varphi(\underline{0}) + x_1 (\partial\varphi/\partial x_1)$. Repeating this application of Hadamard for the other variables yields that the class of φ is determined by $a_\phi = \varphi(\underline{0})$ together with all iterated partial derivatives at $\underline{0}$ without repetitions,

$$a_H = \frac{\partial \varphi}{\partial H}(\underline{0}) = \frac{\partial \varphi}{\partial x_{i_1} \cdots \partial x_{i_k}}(\underline{0}), \text{ where } H = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}.$$

So φ is equivalent to the map $\hat{\varphi}: \mathbb{R}^m \to \mathbb{R}$ defined by

$$(x_1, \ldots, x_m) \longmapsto \textstyle\sum_{H \subseteq \{1, \ldots, m\}} a_H \cdot \prod_{i \in H} x_i.$$

(In fact, every map $\varphi: D^m \to \overline{C^\infty(\mathbb{R})}$ has a unique representation by a $\hat{\varphi}: \mathbb{R}^m \to \mathbb{R}$ of this form.) Let us now look at the defining condition that φ, or $\hat{\varphi}$, be a map of $\mathcal{F}$ (this condition does not depend on the representant of an equivalence class):

$$\text{if } f: \mathbb{R} \to \mathbb{R} \in (y^2) \text{ then also } f \circ \hat{\varphi} \in (x_1^2, \ldots, x_m^2).$$

Equivalently, that $\hat{\varphi}^2 \in (x_1^2, \ldots, x_m^2)$. But modulo $(x_1^2, \ldots, x_m^2)$, $\hat{\varphi}^2$ is the function

$$(x_1, \ldots, x_m) \longmapsto \sum \{a_{H_1} \cdot a_{H_2} \cdot \textstyle\prod_{i \in H_1 \cup H_2} x_i \mid H_1, H_2 \text{ disjoint } \subseteq \{1, \ldots, m\}\}$$

which must therefore be the zerofunction. So by fixing $H \subseteq \{1, \ldots, m\}$ and choosing $x_i = 0$ if $i \notin H$, $x_i = 1$ if $i \in H$, we find that $\sum \{a_{H_1} \cdot a_{H_2} | H_1, H_2$ disjoint, $H_1 \cup H_2 = H\} = 0$, which is precisely the classification theorem (2), provided we consider the condition $a_\phi = 0$ separately, and note the fact that a map $D^m \to D^n$ is an n-tuple of such maps $\varphi : D^m \to D$. Finally, let us look at result (4). A monad on T consists of two maps $\mu : T^2 \to T$ and $\eta : T^0 = \mathrm{Id} \to T$ in the tangent functor category, such that $\mu \circ T\mu = \mu \circ \mu T$ and $\mu \circ T\eta = \mathrm{Id} = \mu \circ \eta T$; or equivalently, two maps $\mu : D \to D^2$ and $\eta : D \to 1$ in $\mathscr{F}$ such that these two diagrams commute

$$\begin{tikzcd} D \arrow[r, "\mu"] \arrow[d, "\mu"'] & D^2 \arrow[d, "\mu \times 1"] \\ D^2 \arrow[r, "1 \times \mu"'] & D^3 \end{tikzcd} \qquad \begin{tikzcd} D \arrow[dr, "\mathrm{id}"'] & D^2 \arrow[l, "1 \times \eta"'] \arrow[r, "\eta \times 1"] & D \arrow[dl, "\mathrm{id}"] \\ & D \arrow[u, "\mu"] & \end{tikzcd}$$

But there is only one such η, and $1 \times \eta$, $\eta \times 1$ are the projections, so from the second diagram μ must be the diagonal map. And indeed, this makes the first diagram commute as well.

CONCLUDING REMARKS

(i) We could also have enlarged $\mathscr{M}$ by a category bigger than $\mathscr{F}$, for example, by adjoining arbitrary intersections so as to obtain duals of C^∞-rings $C^\infty(\mathbb{R}^n)/I$ where I is now an arbitrary ideal. The full embedding of $\mathscr{M}$ into this category, call it $\mathscr{F}'$, is then more easily described: a manifold $M \subset \mathbb{R}^n$ corresponds to the dual of $C^\infty(\mathbb{R}^n)/z(M)$, $z(M)$ being the ideal of functions vanishing on M. This bigger category contains all the duals $\bar{W}$ of Weil-algebras W (so in particular, all the jets of Ehresmann), and an analogous description of the category of natural transformations $T_{\bar{W}}^n \to T_{\bar{W}}^m$ can be given, where $T_{\bar{W}}$ is the functor assigning to a manifold M the bundle $M^{\bar{W}}$. (Such generalizations of the tangent bundle are called prolongations, and were introduced in Weil (1953).) The group of automorphisms of the iterated prolongation functor $T_{\bar{W}}^n$ can be analysed in a similar way as that of the iterated tangent functor T^n, and one can conclude that

(1) $\qquad \mathrm{Aut}(T_{\bar{W}}^n) \cong \mathrm{Aut}(\bar{W}^n).$

A. Kock pointed out to us that this group (1) is also a Lie-group. Indeed, going back to the original Weil-algebra W, we have to show that $\mathrm{Aut}(W \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} W)$ is a Lie-group (n-fold tensor product).

Suppose in general that Γ is a bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $n \times n^2$-matrix γ_{ij}^k, and let $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$ be a linear map with $n \times n$-matrix α_q^p. Then A is a homomorphism w.r.t. the binary operation Γ if for all i, j and l,

$$\sum_k \gamma_{ij}^k \alpha_k^l = \sum_{k,m} \alpha_i^k \alpha_j^m \gamma_{km}^l. \tag{2}$$

In particular, if Γ provides $\mathbb{R}^n$ with a Weil-algebra structure with unit $e_1 = (1, 0, \ldots, 0)$, then the group of automorphisms of this Weil-algebra – call it W – is the subgroup of $GL(n, \mathbb{R})$ carved out by the equations (2) and the additional equation $\alpha_1^i = \delta_{1i}$ (Kronecker δ). Consequently, this group Aut(W) is a Lie-group (by the well-known fact that any subgroup of $GL(n, \mathbb{R})$ carved out by equations in the n^2 entries is a submanifold, so in particular a Lie-group).

Exactly the same argument applies to Aut($W \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} W$).

(ii) We should note here that categories of generalized manifolds like $\mathscr{F}$ and $\mathscr{F}'$ described above are not new. In fact, it precisely such extensions of the category of manifolds that are systematically studied in synthetic differential geometry (see Kock (1981) and references cited there).

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